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THE TRANSPORT OF CHEMICALLY REACTING SPECIES IN THE CLASSICAL CAPILLARY

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SUMMARY

A simplified mathematical model of the transport of chemically reacting species in the microcirculatory system is presented. The differential equation describing mass transport in the capillary is derived. An order of magnitude analysis of the individual terms in the diffusion equation leads to some simplifications and to the consideration of two principal regimes of interest which are defined in terms of a parameter that arises from the differential equations. The parameter suggests some "larger scaled" experimental models. A number of closed-form solutions of the differential equations are presented for special cases. The "general" case is discussed and a means for studying it is suggested.

INTRODUCTION

It is well known that in the microcirculatory system there are various types and sizes of capillaries and that, correspondingly, the manner in which cells and plasma pass through these capillaries varies considerably (drawings of four types of muscle capillaries in guinea pigs obtained from ref. 1 are shown in ref. 2).

Generally, cells pass through capillaries in single file and are deformed by the capillary wall. In some instances the capillary wall is also deformed noticeably, but in others it is not. It has been observed that under some circumstances, cells trail one another by an appreciable distance through the capillary, while under other circumstances the cells move through the capillary butted against one another forming an almost continuous chain. It is the latter continuous chain of cells moving through a capillary that will be called the classical capillary and is the subject of this study.

It is reasonable to expect that chemical species are exchanged between blood and tissues by a variety of mechanisms. Transport of chemical species may be accomplished by convection (i.e., by bulk flow associated with the mass motion of the cells and fluids), molecular diffusion relative to the mass motion, or other phenomena. Transport by an interesting convective flow pattern in the plasma separating two cells has recently been investigated and reported in references 2 and 3, while that by diffusion (including diffusion through tissues) has been examined and reported in references 4 through 8.

The present analysis considers the transport of species i by both the mass motion of the train of cells and by diffusion relative to the mass motion. It includes the exchange of chemical species through a semipermeable wall separating the cellular fluid and the surroundings. These surroundings may be either an annulus of plasma around the cells or whatever exists outside the capillary wall, provided the cell wall and capillary wall are in sufficiently close contact that they can be lumped together as a single semipermeable barrier. Although it is recognized that in ordinary circumstances chemical reactions do not occur to a significant extent in the cells, such reactions are included in the theory so that the results may apply as well to extraordinary circumstances in which chemical changes do occur (e.g., from the introduction of foreign matter by inhalation - N_2O, CO_2, or CO, or by injections into either the blood or tissue). Both the steady and unsteady states are included in the analysis.

It will become apparent that the mathematical model employed to study the phenomena described above has been simplified to a considerable extent. Many of the details known about the microcirculation have been omitted so that solutions of the governing equations may be obtained. In spite of the simplifications, it is expected that the results obtained will provide a tool for evaluating experimental evidence, will classify a number of important regimes, will provide convenient scaling laws, and, hopefully, will permit meaningful quantitative prediction of the phenomena in question.

SYMBOLS[1]

c_i	mass fraction of species i (eq. (4))
D_i	binary diffusion coefficient of species i (dimensional quantity)
f	dimensional mass flux per unit length of capillary leaving the semipermeable wall
F	defined by equation (31)
g(y)	initial or starting profile of species concentration (eq. (18))
h_i	wall permeability for species i
H_i	defined by equation (13)
I_p	modified Bessel function of first kind of order p
J_p	ordinary Bessel function of first kind of order p
L	capillary length (dimensional quantity)
M_i	molecular weight of species i

[1]All quantities are dimensionless unless noted otherwise.

p^2 a constant or constant of proportionality in the production rate

$\vec{q}$ mass averaged velocity vector (eq. (2))(dimensional quantity)

$\vec{q}_i$ diffusive velocity vector relative to mass averaged velocity (eq. (1)) (dimensional quantity)

r radial distance in capillary

R radius of semipermeable wall (dimensional quantity)

s represents t or x for $H_i^2 \geq O[1]$ unsteady state or $H_i^2 = O[\epsilon]$ steady state, respectively

t time variable

u component of mass averaged velocity parallel to capillary axis

v component of mass averaged velocity in radial direction

x distance along capillary axis

y defined by equation (16)

Z particular solution of nonhomogeneous differential equation with nonhomogeneous boundary conditions (eq. (31), e.g.)

α_i zero for constant flux of species i across semipermeable wall; otherwise it is $-H_i h_i$

β_i $H_i h_i (c_{i_c} - c_{i_s})$ for constant flux of species i across semipermeable wall; otherwise it is $-H_i h_i c_{i_s}$

ϵ a quantity much less than unity

λ_n eigenvalues obtained from positive roots of equation (30)

ρ mass density of solution

ρ_i mass density of species i

ω_i local rate of production of species i per unit volume

$(\bar{\ })$ dimensional quantities

Subscripts

c conditions on the cell side (just inside) of the semipermeable wall

i,j,k species i, j, k

n nth eigenvalue

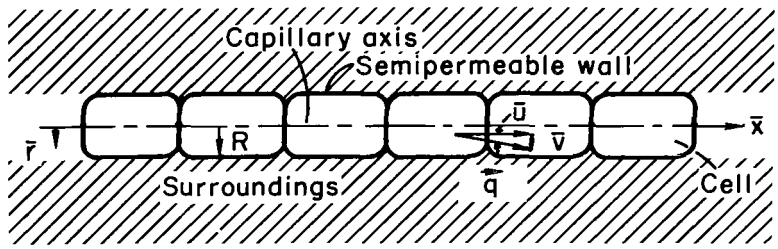
o reference conditions (conditions on axis of capillary entrance)

s conditions on surroundings side (just outside) of the semipermeable wall

ANALYSIS

Coordinates and Kinematics

The flow model used in the analysis is shown in the sketch. The cells contain a primary fluid and quantities of various chemical species i, which are transported by the motion of the cell and by diffusion. Thus each chemical species moves with an absolute velocity $\vec{V}_i$ composed of the local mass averaged velocity $\vec{q}$ plus its diffusion velocity $\vec{q}_i$ relative to the mass averaged velocity; that is



$$\vec{V}_i = \vec{q} + \vec{q}_i \tag{1}$$

The mass averaged velocity has components $\bar{u}$ and $\bar{v}$ parallel and normal to the capillary axis, respectively, and is defined by

$$\vec{q} = \frac{\sum \bar{\rho}_i \vec{V}_i}{\sum \bar{\rho}_i} = \sum \bar{c}_i \vec{V}_i \tag{2}$$

where

$$\bar{\rho} = \sum_i \bar{\rho}_i \tag{3}$$

and the mass fraction of species i is

$$\bar{c}_i = \bar{\rho}_i / \bar{\rho} \tag{4}$$

It follows simply that

$$\sum \vec{q}_i \bar{c}_i = 0 \tag{5}$$

throughout the flow field.

The mass flux of species i by diffusion relative to the mass motion is the product of species density and diffusion velocity and, according to Fick's law, is

$$\bar{\rho}_i \vec{q}_i = -\bar{\rho} D_i \text{ grad } \bar{c}_i \tag{6}$$

For the model of blood cells being pushed along, one behind the other through the capillary, one cannot expect to write the usual Navier-Stokes momentum equation and then look for the Poiseuille pipe flow velocity profiles. Instead, it is simply noted that the friction forces between the cell wall and the surroundings must be balanced with a pressure gradient in order to maintain the flow through the capillary. For present purposes, we are not concerned with any more detail of momentum balancing but consider mass transport of the species i.

Differential Equation of Mass Transport

The mass transport in the capillary is accomplished both by movement of the cells with their chemical diluents and by diffusion of the diluents relative to the cell motion.

In a fixed mass element of fluid moving with mass averaged velocity, we equate the rate at which the concentration of species i varies (left side) to the net rate at which it diffuses into the element and is produced therein (right side) leading to

$$\bar{\rho} \frac{D\bar{c}_i}{D\bar{t}} = \frac{\partial}{\partial \bar{x}} \left(\bar{\rho} D_i \frac{\partial \bar{c}_i}{\partial \bar{x}} \right) + \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left(\bar{r} \bar{\rho} D_i \frac{\partial \bar{c}_i}{\partial \bar{r}} \right) + \bar{\omega}_i \tag{7}$$

The left side of equation (7) contains the Eulerian derivative $D/D\bar{t}$ which is discussed on page 15 of reference 9. The mass flux of species i by diffusion represented by equation (6) is readily identified in the first two terms of the right side of equation (7), while the last term is the local rate of production of species i per unit volume.

It is important to note that as long as we are concerned with dilute quantities of species i diffusing in some primary fluid, the binary diffusion coefficient D_i may be used. Furthermore, for a mixture of several species which are not necessarily dilute we may use the binary diffusion coefficient if it is possible to combine the various species into two principal groups, each containing species of like molecular weight and like mutual collision cross section (ref. 10). Otherwise, it is necessary to use the multicomponent diffusion coefficients, which are most unpleasant. The boundary conditions for equation (7)[2] are

[2]Boundary conditions in the x direction imposed by the boundaries of the end walls of the cells have not been written. It will be shown subsequently that for the model employed, these boundaries are not important.

at $\bar{t} = 0$, $$\bar{c}_i = \bar{c}_i(\bar{x},\bar{r},0) \tag{8}$$

at $\bar{x} = 0$, $$\bar{c}_i = \bar{c}_i(0,\bar{r},\bar{t}) \tag{9}$$

at $\bar{r} = 0$, $$\partial\bar{c}_i/\partial\bar{r} = 0 \tag{10}$$

at $\bar{r} = R$, $$-D_i \frac{\partial\bar{c}_i}{\partial\bar{r}} = \bar{h}_i(\bar{c}_{i_c} - \bar{c}_{i_s}) \tag{11}$$

It is convenient to define the dimensionless quantities

$$\left.\begin{aligned} x &= \bar{x}/R\ , & r &= \bar{r}/R\ , & \rho &= \bar{\rho}/\bar{\rho}_o \\ c_i &= \bar{c}_i/\bar{c}_{i_o}\ , & u &= \bar{u}/\bar{u}_o\ , & v &= \bar{v}/\bar{u}_o \\ t &= \frac{\bar{u}_o\bar{t}}{R}\ , & h_i &= \bar{h}_iR/D_i\ , & \omega_i &= \bar{\omega}_iR/\bar{c}_{i_o}\bar{u}_o\bar{\rho}_o \end{aligned}\right\} \tag{12}$$

Define also the dimensionless parameter

$$H_i^2 = D_i/R\bar{u}_o \tag{13}$$

Substituting these dimensionless quantities into equation (7) and expressing the Eulerian derivative leads to:

$$\frac{\partial c_i}{\partial t} + u\frac{\partial c_i}{\partial x} + v\frac{\partial c_i}{\partial r} = H_i^2\left(\frac{\partial^2 c_i}{\partial x^2} + \frac{1}{\rho}\frac{\partial c_i}{\partial x}\frac{\partial\rho}{\partial x} + \frac{1}{r}\frac{\partial c_i}{\partial r} + \frac{\partial^2 c_i}{\partial r^2} + \frac{1}{\rho}\frac{\partial c_i}{\partial r}\frac{\partial\rho}{\partial r}\right) + \frac{\omega_i}{\rho} \tag{14}$$

The size of the individual terms in equation (14) is estimated as follows (where $\epsilon \ll 1$, L is capillary length, and $R/L = O[\epsilon]$).

$$\frac{\partial c_i}{\partial t} + 1\,\frac{1}{\epsilon^{-1}} + \epsilon^2\,\frac{1}{1} = H_i^2\left(\frac{1}{\epsilon^{-2}} + \frac{1}{1}\frac{1}{\epsilon^{-1}}\frac{\epsilon}{\epsilon^{-1}} + \frac{1}{1}\frac{1}{1} + \frac{1}{1} + \frac{1}{1}\frac{1}{1}\frac{\epsilon}{1}\right) + \frac{\omega_i}{\rho} \tag{15}$$

On the left side of equation (15) ϵ^2 may be neglected in favor of ϵ, and in the parentheses, all terms ϵ and smaller may be neglected in comparison with unity. If u is set to unity (in accord with the model employed), r is replaced by

$$y = r/H_i \tag{16}$$

and the negligible terms are omitted, equation (14) becomes

$$\frac{\partial c_i}{\partial t} + \frac{\partial c_i}{\partial x} = \frac{\partial^2 c_i}{\partial y^2} + \frac{1}{y}\frac{\partial c_i}{\partial y} + \frac{\omega_i}{\rho} \tag{17}$$

Examination of equation (15) reveals two regimes of interest which depend on the size of H_i^2. The first of these corresponds to $H_i^2 = 0[\epsilon]$, for which equation (17) is appropriate as it stands. The second of these corresponds to $H_i^2 \geq 0[1]$ for which the second term on the left of equation (17) may be omitted. The boundary conditions for equation (17) are:

for $H_i^2 \ll 1$, steady state, or $H_i^2 \geq 0[1]$,

$$c_i(0,y) = g(y) \tag{18}$$

$$\frac{\partial c_i}{\partial y}(x,0) = 0 \quad \text{or} \quad \frac{\partial c_i}{\partial y}(t,0) = 0 \tag{19}$$

$$\frac{\partial c_i}{\partial y}\left(x, \frac{1}{H_i}\right) = -H_i h_i (c_{i_c} - c_{i_s}) \quad \text{or} \quad \frac{\partial c_i}{\partial y}\left(t, \frac{1}{H_i}\right) = -H_i h_i (c_{i_c} - c_{i_s}) \tag{20}$$

It is convenient to let s represent x for $H_i^2 \ll 1$ steady state or t for $H_i^2 \geq 0[1]$ and write boundary condition (20) in the form:

$$\frac{\partial c_i}{\partial y}\left(s, \frac{1}{H_i}\right) = \alpha_i c_i \left(s, \frac{1}{H_i}\right) - \beta_i \tag{21}$$

where α_i and β_i are considered to be constant. Equation (21) is related to the mass flux of species i across the semipermeable wall.[3] If the flux is a function of s, then by comparison of equations (21) and (20), and equations (12)

$$\alpha_i = -H_i h_i = -\frac{\bar{h}_i}{\bar{u}_o H_i}, \quad \beta_i = -H_i h_i c_{i_s} = -\frac{\bar{h}_i}{\bar{u}_o H_i}\frac{\bar{c}_{i_s}}{\bar{c}_{i_o}} \tag{22}$$

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[3]The mass flux per unit length of capillary leaving the semipermeable wall is

$$f = -2\pi R \bar{\rho} D_i \frac{\partial \bar{c}_i}{\partial \bar{r}} = -2\pi R \bar{u}_o \bar{c}_{i_o} \bar{\rho}_o \frac{\partial c_i}{\partial y}$$

or in terms of wall permeability is

$$f = 2\pi R \bar{h}_i \bar{\rho}(\bar{c}_{i_c} - \bar{c}_{i_s}) = -2\pi R \bar{u}_o \bar{\rho}_o \bar{c}_{i_o} \rho(\alpha_i c_{i_c} - \beta_i)$$

On the other hand, if the flux of species i across the semipermeable membrane is constant, we must set α_i to zero and let β_i stand for

$$\beta_i = H_i h_i (c_{i_c} - c_{i_s}) \tag{23}$$

Both of these possibilities are included in boundary condition (21).

It is noted on the right side of equation (17) that diffusion in the axial direction has been neglected in favor of diffusion in the radial direction because the concentration gradients in the former are smaller. Thus even though there are cell walls inhibiting diffusion in the x direction, we can neglect that detail simply because diffusion with or without cell walls is not important in the x direction. The point made earlier accounting for the cell wall by including its permeability with that of the capillary wall (which thus associates the importance of cell wall with radial diffusion) is a better approximation than may have been anticipated.

Without having solved the equations, we already have a useful criterion, H_i^2, which delineates two important regimes. These will be called the bulk flow and diffusion controlling regimes corresponding to $H_i^2 \ll 1$ and $H_i^2 \geq O[1]$, respectively. The significance of the criterion $H_i^2 \ll 1$ for differential equation (17) is that mass transport of species i by convection (or bulk flow) in the axial direction is comparable to mass transport by diffusion in the radial direction.

The parameter H_i may have a fairly broad range of values. For biological systems, $D_i \approx 10^{-7}$ cm^2/sec for several metabolites, according to reference 7 (p. 13). That reference cites the case of the diffusion of oxygen in Arbacia eggs (p. 80). The diffusion coefficient for glycerine, $MgSO_4$, KCl, NaCl, sugar, and urea in water is about two orders of magnitude larger than that (i.e., 10^{-5} cm^2/sec) according to reference 11 (p. 210). For human capillaries, $R \approx 0.4 \times 10^{-3}$ cm, while $\bar{u}_o \approx 0.04$ cm/sec according to reference 12 (p. 45). Both values are almost the same as those listed for dogs (ref. 13, p. 43)(for these conditions, the time required for a cell to travel the length of the capillary is about one second). Thus H_i varies from about the order of 10^{-2} to unity corresponding to these values.

Under some circumstances, the diffusion coefficient in gases and liquids is proportional to the reciprocal of the square root of the molecular weight of the diffusing substance (ref. 14, p. 171). Noting that the square root of the molecular weight of thiocyanate is the same order of magnitude as that for oxygen or other metabolites, we might expect H_i^2 for thiocyanate to be of the order of 10^{-2}. For this reason, if thiocyanate is introduced into the red cells, it may be expected that its transport by axial bulk flow is significant. The experimental results (in dogs) of reference 15 indicate that indeed this is the case (although the mechanism for the escape of thiocyanate from the capillary was thought to be "filtration in bulk" through the capillary wall which was considered to be more rapid than diffusion); roughly one fourth of the cardiac output of ultrafiltrable material entering the capillary traversed the length of the capillary without escaping to the surroundings.

On the other hand, the significance of the criterion $H_i^2 \geq O[1]$ is that mass transport by radial diffusion is more important than that by the capillary flow. This may be the case if D_i is large (corresponding to species having low molecular weights) or if $R\bar{u}_o$ is small (corresponding to capillaries in which the flow is almost cut off, i.e., only a trickle of blood flows through).

The formulation of the differential equation (17) and its boundary conditions (18), (19), and (21) is complete and we turn attention to solutions of the equations.

RESULTS AND DISCUSSION

If $\bar{\omega}_i/\bar{\rho}$, the local rate of production or consumption of species i per unit mass, is either constant (such as zero for no reactions) or proportional to $\bar{c}_i$ (which is likely for consumption), the differential equation (17) is linear and one may expect to find solutions in a direct way. Generally, however, $\bar{\omega}_i/\bar{\rho}$ is not necessarily constant or proportional to $\bar{c}_i$, but may be proportional to the mass fraction of some other species $\bar{c}_j$ (if $\bar{c}_i$ is formed by the dissociation of $\bar{c}_j$ for example) as well as to the mass fraction of some enzyme $\bar{c}_k$. In general, an equation of the type (17) can be written for each species present, and the resulting set of equations may be both nonlinear and coupled. Moreover, boundary conditions on these equations may not be well behaved, especially for studies in which the capillary wall has been damaged.

Our first interest is to examine the linear case for which solutions are readily obtained. They will be presented in order of increasing generality. Subsequently, a means for studying the nonlinear case will be discussed.

Closed-Form Solutions

These results are presented in order of increasing generality.

Steady-state, diffusion-controlling, constant production or consumption rate ($H_i^2 \geq O[1]$, $\omega_i/\rho = \pm p^2$). - The left side of equation (17) is set to zero and its solution is the parabola

$$c_i(y) = \mp\left(\frac{p}{2}\right)^2 y^2 \pm \frac{p^2}{2H_i}\left(\frac{1}{2H_i} - \frac{1}{\alpha_i}\right) + \frac{\beta_i}{\alpha_i} \tag{24}$$

Illustrative examples of this result are presented in figures 1 (for production) and 2 (for consumption). The parameters p^2, H_i, β_i, and α_i shown were selected arbitrarily. The slopes of c_i at the semipermeable wall indicate that species i is leaving the capillary during constant production and entering during constant consumption.

Two special cases corresponding to $\alpha_i = 0$ (constant mass flux at the semipermeable wall) and $p^2 = 0$ (no reactions) might be mentioned. For $\alpha_i = 0$ the solution of equation (17) becomes

$$c_i(y) = + \frac{p^2}{4} \left(\frac{1}{H_i^2} - y^2 \right) \tag{25}$$

It is noted that only the production of species i ($+p^2$) is allowed because c_i cannot be negative and that species i vanishes at the wall (c_{i_c} is zero). Furthermore, it is easy to show that $\beta_i = + p^2/2H_i$, which means that species i is leaving the capillary (eq. (23)). But if c_{i_c} is zero, species i cannot leave the capillary, so we must require that p^2 be zero, and the case is trivial.

The second special case corresponds to no reactions ($p^2 = 0$), but with $\alpha_i \neq 0$. The result is that $c_i = \beta_i/\alpha_i$, a constant, and that mass transport across the semipermeable wall is zero. Thus it is seen that equation (24) is meaningful only if there are chemical reactions.

Steady-state, diffusion-controlling, production or consumption rate proportional to concentration ($H_i^2 \geq 0[1]$, $\omega_i/\rho = \pm p^2 c_i$).- Again the left side of equation (17) is omitted and its solution for production is

$$c_i(y) = \frac{\beta_i}{\alpha_i J_0 \left(\frac{p}{H_i} \right) + p J_1 \left(\frac{p}{H_i} \right)} J_0(py) \tag{26}$$

while that for consumption is

$$c_i(y) = \frac{\beta_i}{\alpha_i I_0 \left(\frac{p}{H_i} \right) - p I_1 \left(\frac{p}{H_i} \right)} I_0(py) \tag{27}$$

Illustrative examples for production and consumption are shown in figures 3 and 4 for the same values of the parameters used in figures 1 and 2. The mass fraction gradients at (and the mass flux across) the wall are smaller in figures 3 and 4 than those in figures 1 and 2, respectively.

As before, equations (26) and (27) are trivial for no production ($p^2 = 0$) and $c_i = \beta_i/\alpha_i$ = constant. For constant mass flux across the wall, equation (27) must be abandoned (c_i is negative), but equation (26) is well behaved. So far, examples are very readily evaluated because c_i has been a function of only one independent variable. Results for two independent variables (s,y) are considered next.

Production and consumption rate constant ($\omega_i/\rho = \pm p^2$) for either unsteady-state diffusion controlling ($H_i^2 \geq 0[1]$) or steady-state bulk flow ($H_i^2 \ll 1$).- Here we let s represent t for the former and x for the latter, while the

second term on the left of equation (17) is neglected for the former, but the first term is neglected for the latter. The solution of equation (17) is obtained by separation of variables for the homogeneous part of the equation (ref. 16, p. 168) to which the appropriate particular solution is added and is:

$$c_i(s,y) = \frac{\beta_i}{\alpha_i} \pm p^2 \left[\frac{1}{H_i} \left(\frac{1}{2H_i} - \frac{1}{\alpha_i} \right) - \frac{y^2}{2} \right] + \sum_{n=1}^{\infty} A_n J_o(\lambda_n y) e^{-\lambda_n^2 s} \tag{28}$$

where

$$A_n = \frac{2\lambda_n^2}{\left[\left(\frac{\lambda_n}{H_i} \right)^2 + \left(\frac{\alpha}{H_i} \right)^2 \right] \left[J_o \left(\frac{\lambda_n}{H_i} \right) \right]^2} \int_o^{1/H_i} y J_o(\lambda_n y) F(y) dy \tag{29}$$

λ_n are the positive roots (presented in the appendix) of the transcendental equation

$$\lambda = -\alpha_i \frac{J_o \left(\frac{\lambda}{H_i} \right)}{J_1 \left(\frac{\lambda}{H_i} \right)} \tag{30}$$

and

$$F(y) = g(y) - Z(y) \tag{31}$$

where $Z(y)$ is the particular solution mentioned previously (which includes the terms before the summation sign on the right side of equation (28) and is seen to be the same as the right side of equation (24)).

Equation (28) can be evaluated for a given set of parameters H_i^2, α_i, β_i, p^2, and $g(y)$. Although the equation is in terms of well-known functions and the procedure is straightforward, its evaluation is tedious and time consuming. Thus it is likely that the equation should be evaluated by a high-speed digital computer for each specific example (which is beyond the scope of the present analysis). For present purposes it is instructive to examine two simplifications of equation (28); the first corresponds to constant mass flux across the semipermeable wall ($\alpha_i = 0$) for which equation (28) becomes

$$c_i(s,y) = \pm \frac{p^2}{4} \left(\frac{1}{H_i^2} - \frac{1}{y^2} \right) + \sum_{n=1}^{\infty} A_n J_o(\lambda_n y) e^{-\lambda_n^2 s} \tag{32}$$

where

$$A_n = 2H_i^2 \int_o^{1/H_i} y J_o(\lambda_n y) F(y) dy \tag{33}$$

and λ_n/H_i are the zeros of J_1 tabulated in reference 17 (p. 166) and are

$$\frac{\lambda_1}{H_i} = 3.8317 , \quad \frac{\lambda_2}{H_i} = 7.0156 , \quad \frac{\lambda_3}{H_i} = 10.1735 , \quad . . . \tag{34}$$

The second simplification is for no chemical reactions ($p^2 = 0$, but $\alpha_i \neq 0$). If we assume that the concentration of species i is uniform in the cell as it enters the capillary, then $g(y) = 1$ and A_n can be integrated readily. The resulting mass fraction equation is much simpler than equation (28) and is

$$c_i(s,y) = \frac{\beta_i}{\alpha_i} + 2H_i(\beta_i - \alpha_i) \sum_{n=1}^{\infty} \frac{1}{(\lambda_n^2 + \alpha^2)} \frac{J_o(\lambda_n y)}{J_o(\lambda_n/H_i)} e^{-\lambda_n^2 s} \tag{35}$$

where, as before, λ_n are the positive roots of equation (30).

Equation (35) holds special significance in that it corresponds to the transport of species i under most ordinary circumstances, and it is appropriate that some illustrative examples obtained from it be presented. The first of these corresponds to steady-state bulk flow. Some values of the parameters α_i and H_i for this condition have been estimated for various species by use of data obtained from reference 18, and are listed in table I. It is pointed out that the following assumptions were made to obtain the values shown in the last two columns $\bar{u}_o = 0.04$ cm/sec, $R = 0.4 \times 10^{-3}$ cm, and $D_i \sqrt{M_i} \approx 10^{-6}$ $cm^6 g^{1/2}/sec(g\ mole)^{1/2}$. It is emphasized that there is considerable uncertainty associated with these numbers and they are only intended for illustrative purposes. It is noted that H_i only varies by an order of magnitude (10^{-1} to 10^{-2}) for species whose molecular weight varies by three orders of magnitude (from 18 for water to 69,000 for serum albumen), and that most of the species of interest can be represented by $H_i \approx 10^{-1}$. Similarly, α_i is of the order of -10^{-1} to -10^{-3} for most of these species, while -10^{-1} represents a "typical" value for illustrative purposes. Equation (22) reveals that β_i can vary from zero (no species i in surroundings) to $-\infty$ (no species i at entrance to capillary). For illustrative purposes we set $\beta_i = 0$ while $H_i = -\alpha_i = 10^{-1}$. This corresponds to cells laden with species i entering a capillary and giving up that species to surroundings that are relatively free of that substance.

The development of the mass fraction profile across the cell as it moves along the capillary (x = 0, 1, 10 and 100) is shown in figure 5. It can be seen that the cell gives up 75 percent of its mass of species i to the surroundings by the time it has passed through a capillary that is 100 radii long (which is a good reason for capillaries to be about that long). Coincidently, this is essentially the result cited previously for thiocyanate injection into dogs (ref. 15) which indicates that the estimate of $H_i^2 = 10^{-2}$ is plausible for thiocyanate, and that at least this result of that paper could be explained by the diffusion mechanism. The rate at which species i leaves the capillary radially varies with position for this example and is shown in figure 6.

The second of these examples corresponds to the unsteady state with diffusion controlling, which is illustrated in figure 7 for $H_i^2 = 1$, $\alpha_i = -0.1$, $\beta_i = p^2 = 0$. For this example, cells laden with species i were moving through the capillary whose surroundings are free of that substance. Abruptly, the flow was retarded or the permeability of the wall to this species was increased. The resulting mass fraction profile of species i across the cell is shown at times t = 0, 1, 10, and 100. For this particular example, the mass fractions are not strong functions of radius, but are strong functions of time. The lack of radial dependence indicates that the adjustment of concentration within the cell is rapid and that the wall permeability is the limiting factor on species transport. This will be apparent subsequently in another unsteady-state diffusion controlling example (fig. 11) as well. Species i essentially vanishes between t = 10 and t = 100. For reference purposes, t = 10 corresponds to 0.1 second if $R = 0.4\times10^{-3}$ cm and $\bar{u}_o = 0.4\times10^{-1}$ cm/sec, or to 1 second if the flow velocity has been diminished to 0.4×10^{-2} cm/sec. Thus in these small vessels, mass transport by diffusion may be accomplished quite rapidly. The rate at which species i leaves the capillary radially is shown as a function of time in figure 8.

Another interesting example without reactions is that of cells moving through a capillary whose surroundings have a high concentration of species i relative to that of the entering cell. In figure 9, we see the concentration of species i increase by a factor of about 8 in a cell as it moves 50 diameters down a capillary whose surroundings have a concentration ten times that of the entering cell. The corresponding mass flux of species i entering the capillary is shown as a function of position in figure 10.

As a final illustration, the development of the concentration profile of species i in the cell in whose surroundings species i abruptly appears in high concentration (by injection for example) is shown in figure 11. The cell concentration rapidly approaches that of the surroundings. Indeed, by the time t = 10 (or 0.1 to 1.0 second for the conditions cited above for fig. 7), the cell concentration is about 90 percent that of the surroundings. The mass flux entering the cell as a function of time is shown in figure 12.

Production or consumption rate proportional to concentration ($\omega_i/\rho = +p^2 c_i$) for either unsteady diffusion controlling ($H_i^2 \geq 0[1]$, or steady-state bulk flow ($H_i^2 \ll 1$).- The solution of equation (17) for production includes the right side of equation (26) as the particular solution Z(y) and is

$$c_i(s,y) = \frac{\beta_i J_o(py)}{\alpha_i J_o\left(\frac{p}{H_i}\right) + pJ_1\left(\frac{p}{H_i}\right)} + \sum_{n=1}^{\infty} A_n J_o(\lambda_n y) e^{(p^2-\lambda_n^2)s} \qquad (36)$$

where

$$A_n = \frac{2H_i^2}{J_o^2\left(\frac{\lambda_n}{H_i}\right)\left(1 + \frac{\alpha^2}{\lambda_n^2}\right)} \int_o^{1/H_i} yF(y)J_o(\lambda_n y)\,dy \qquad (37)$$

The function $F(y)$ is expressed by equation (31) and λ_n are the positive roots of equation (30).

In similar fashion, the solution of equation (17) for consumption includes the right side of equation (27) as $Z(y)$ and is

$$c_i(s,y) = \frac{\beta_i I_o(py)}{\alpha_i I_o\left(\frac{p}{H}\right) - pI_1\left(\frac{p}{H}\right)} + \sum_{n=1}^{\infty} A_n J_o(\lambda_n y) e^{-(p^2+\lambda_n^2)s} \qquad (38)$$

and equations (30), (31), and (37) apply. As was the case in the previous section, some simplification of both equations (36) and (38) is effected if either α_i in equation (30) or p^2 in equations (36) and (38) is set to zero. The first of these ($\alpha_i = 0$) leads to eigenvalues corresponding to the zeros of J_1, as before. Similarly, if p^2 is zero, both equations (36) and (38) reduce to equation (35) which has already been discussed.

The two-dimensional (s and y) results of the last two sections may be applied to many problems. Two prominent unsteady-state problems which suggest themselves are: (1) effect of a sudden appearance locally of a particular chemical species (by injection for example), and (2) species transport following the abrupt termination of flow[4] (caused, perhaps, by a leukocyte clogging a capillary).

Obviously, the steady-state results may be applied to any number of examples. Not quite so obvious, perhaps, is that to some extent, they may be applied directly to the case of a wall of discontinuous permeability (resulting from damage for example). That is, they can be applied to the damaged length using the results from the undamaged length to obtain $g(y)$ at the beginning of the damaged length.

Closed-form solutions for the three-dimensional case (x,y,t)(or unsteady state) could be obtained to complete the catalog of results of coupled linear differential equations.

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[4]In spite of the fact that $H_i^2 = \infty$ when $\bar{u}_o = 0$, the present results can be used if we redefine some of the quantities. In particular, we replace $\bar{u}_o$ by D_i/R wherever it appears in the formulation. The consequence is that H_i^2 can be replaced by unity everywhere it appears and the following quantities have these new definitions:

$$t \equiv \bar{t}\,\frac{D_i}{R^2}, \quad \omega_i \equiv \bar{\omega}_i \frac{R^2}{\bar{\rho}_o \bar{c}_{i_o} D_i}, \quad \text{and either} \quad \alpha_i \equiv -h_i = -\bar{h}_i \frac{R}{D_i},$$

$$\beta_i = -h_i c_{i_s} \quad \text{or} \quad \alpha_i = 0, \quad \beta_i = h_i(c_{i_c} - c_{i_s})$$

Solution of the General Case

It was mentioned previously that, in general, the term ω_i/ρ accounting for the local rate of production of the species i per unit mass, will be neither constant nor proportional to c_i; but may be complicated function of c_i and the concentration of other species. Furthermore, the boundary conditions may not be well behaved. Thus it is likely that numerical integration of the partial differential equations will be necessary.

Examination of equation (17) suggests a numerical method for solving a set of equations (17) corresponding to a set of reacting chemical species for either two-dimensional case (y and t or y and x). The equation is a parabolic differential equation. If profiles of $c_i(y,s)$ are known at some value of s (from the results of an exact solution for example), it is possible to calculate the profile of c_i at $s + \Delta s$ by use of the differential equation itself if the derivatives appearing in the equation are replaced by finite differences corresponding to a rectangular finite difference mesh in y and s. In this way, profiles of $c_i(y,s)$ can be constructed at successive values of s to obtain the complete solution of a problem. The boundary conditions do not have to be particularly well behaved in order for a finite difference scheme to be successful (refs. 19 through 23). They can be varied in an almost arbitrary way and can include discontinuities (corresponding, e.g., to capillary wall injury or to the sudden appearance of a toxic substance). For the general case, involving a number of reacting species, it is expected that this approach would be very fruitful.

Experimental Considerations

The results of the analysis are contained in the closed-form solutions written in terms of the parameters H_i^2, α_i, β_i, $g(y)$, and p^2 and give the mass fraction of species i as a function of time and position. The parameters constitute a set of scaling laws for the reacting flow system. Thus an experiment with a large-scale model represents the behavior of the small-scale prototype (capillary), as long as these parameters are the same in both systems. It is further noted that the experimental model could consist of one long cylindrical "cell" with semipermeable walls and filled with a test fluid being drawn through a concentric tube. The point is that it is not necessary to have numerous individual cells moving in line through the tube because the end walls of individual cells are not important to the transport phenomena. The "surroundings" mentioned in the introduction can be either an annulus of fluid or the concentric fixed semipermeable capillary wall. Furthermore, a number of these tubes may be placed in a bath of suitable "tissue fluid" to represent a capillary bed. Measurements might be made in individual tubes or in the entrance and exit of the complete capillary bed. Thus, the analysis indicates that meaningful experiments can be made with relatively simple macroscopic models using reacting mixtures and semipermeable tubes.

CONCLUSIONS

The differential equation of mass transport of chemically reacting species by diffusion in cells that are moving through capillaries has been simplified by an order of magnitude analysis. Two principal regimes of interest have been defined by the ratio of the binary diffusion coefficient of a given chemical substance to the product of capillary radius and cell velocity. Species transport by the bulk motion of cells along the capillary axis can be neglected in favor of radial diffusion only if that ratio is of the order unity or greater.

A family of closed-form solutions of the differential equations has been obtained and includes steady and unsteady states, and consumption or production of species (with particular reference to the introduction of foreign substances) at a local rate that is either constant or proportional to the local concentration.

Application of these results to microcirculation problems has been discussed and a few illustrative examples were presented showing the development of the concentration profiles across the cell either as a function of time or position as the cell moves through the capillary. Examples of the rate of transport of chemical species in and out of the capillary were presented.

A method of solving more general examples which are described in terms of a set of coupled nonlinear partial differential equations has been discussed.

The analysis suggests scaling parameters useful for modeling experiments of reacting flows passing through vessels with semipermeable walls.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., Aug. 9, 1963

APPENDIX

EIGENVALUES

The positive roots of the transcendental equation (30) provide the eigenvalues λ_n for several of the cases considered. Because that equation appears in problems other than the present application, its roots are of general interest and will be presented for two values of the parameter α_i/H_i.

Because the zeros of J_0 and J_1 interlace and, for large arguments, are spaced at intervals of approximately π, the roots of equation (30) may be expected to have that spacing for large values of λ_n/H_i. For small arguments, the first two roots may be estimated by use of the series representations for J_0 and J_1. If terms through the fourth and third degrees in λ_n/H_i for J_0 and J_1, respectively, are substituted in equation (30), the two positive roots are readily found to be

$$\frac{\lambda_1}{H_i} = 2\sqrt{2}\left(\frac{1}{1 - 4\,\frac{H_i}{\alpha_i}}\right)^{1/2} \tag{A1}$$

and

$$\frac{\lambda_2}{H_i} = 2\sqrt{2} \tag{A2}$$

For examples of present interest, α_i is a negative number (eq. (22)) and the first root (A1) is real and smaller than the second root (A2). It was noted in the text that for $\alpha_i = 0$, the roots of equation (30) are the zeros of J_1, the first of which is zero (which is in accord with (A1)) and the second is 3.8317 (which does not agree with eq. (A2)). For α_i different from zero, equation (A1) predicts the first root fairly accurately, but equation (A2) generally does not predict the second root accurately.

Exact roots of equation (30) have been obtained by Messrs. Paul Byrd and L. Klosinski of Ames Research Center by use of the IBM 7090 digital computer. The first 200 roots λ_n/H_i for $\alpha_i/H_i = -0.1$ and -1.0 are listed in tables II and III, respectively.

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TABLE I.- PARAMETER ESTIMATES

Species i	M_i (g/g mole) (ref. 18)	$\bar{h}_i$ (cm/sec) (ref. 18)	H_i	$-\alpha_i$
Water	18	0.54×10^{-3}	1.2×10^{-1}	1.1×10^{-1}
NaCl	58.5	$.33\times10^{-3}$	$.90\times10^{-1}$	$.91\times10^{-1}$
Urea	60	$.26\times10^{-3}$	$.90\times10^{-1}$	$.72\times10^{-1}$
Glucose	180	$.9\times10^{-4}$	$.68\times10^{-1}$	$.33\times10^{-1}$
Sucrose	342	$.5\times10^{-4}$	$.58\times10^{-1}$	$.22\times10^{-1}$
Raffinose	594	$.39\times10^{-4}$	$.51\times10^{-1}$	$.19\times10^{-1}$
Insulin	5,100	$.50\times10^{-5}$	$.30\times10^{-1}$	$.42\times10^{-2}$
Myoglobin	17,000	$.40\times10^{-6}$	$.22\times10^{-1}$	$.46\times10^{-3}$
Hemoglobin	68,000	---	$.15\times10^{-1}$	---
Serum albumen	69,000	---	$.15\times10^{-1}$	---

TABLE II.- THE ROOTS λ_n/H_i FOR $\alpha/H_i = -0.1$

n	λ_n/H_i	n	λ_n/H_i	n	λ_n/H_i	n	λ_n/H_i
1	0.44168178	51	0.15786328+3	101	0.31494378+3	151	0.47202371+3
2	.38577098+1	52	.16100491+3	102	.31808539+3	152	.47516531+3
3	.70298252+1	53	.16414654+3	103	.32122698+3	153	.47830690+3
4	.10183293+2	54	.16728816+3	104	.32436859+3	154	.48144850+3
5	.13331195+2	55	.17042978+3	105	.32751019+3	155	.48459010+3
6	.16476700+2	56	.17357141+3	106	.33065179+3	156	.48773169+3
7	.19620955+2	57	.17671303+3	107	.33379339+3	157	.49087329+3
8	.22764477+2	58	.17985465+3	108	.33693499+3	158	.49401488+3
9	.25907532+2	59	.18299627+3	109	.34007660+3	159	.49715648+3
10	.29050270+2	60	.18613788+3	110	.34321819+3	160	.50029808+3
11	.32192786+2	61	.18927950+3	111	.34635979+3	161	.50343967+3
12	.35335137+2	62	.19242112+3	112	.34950139+3	162	.50658127+3
13	.38477365+2	63	.19556274+3	113	.35264299+3	163	.50972287+3
14	.41619497+2	64	.19870435+3	114	.35578459+3	164	.51286446+3
15	.44761553+2	65	.20184596+3	115	.35892618+3	165	.51600605+3
16	.47903548+2	66	.20498758+3	116	.36206779+3	166	.51914766+3
17	.51045494+2	67	.20812919+3	117	.36520939+3	167	.52228925+3
18	.54187398+2	68	.21127080+3	118	.36835099+3	168	.52543084+3
19	.57329269+2	69	.21441241+3	119	.37149259+3	169	.52857244+3
20	.60471111+2	70	.21755403+3	120	.37463418+3	170	.53171404+3
21	.63612928+2	71	.22069564+3	121	.37777579+3	171	.53485563+3
22	.66754723+2	72	.22383724+3	122	.38091739+3	172	.53799723+3
23	.69896501+2	73	.22697885+3	123	.38405898+3	173	.54113881+3
24	.73038264+2	74	.23012047+3	124	.38720058+3	174	.54428042+3
25	.76180013+2	75	.23326207+3	125	.39034218+3	175	.54742201+3
26	.79321747+2	76	.23640368+3	126	.39348377+3	176	.55056360+3
27	.82463471+2	77	.23954529+3	127	.39662538+3	177	.55370520+3
28	.85605187+2	78	.24268690+3	128	.39976697+3	178	.55684680+3
29	.88746893+2	79	.24582850+3	129	.40290857+3	179	.55998839+3
30	.91888592+2	80	.24897011+3	130	.40605018+3	180	.56312999+3
31	.95030284+2	81	.25211172+3	131	.40919177+3	181	.56627158+3
32	.98171968+2	82	.25525332+3	132	.41233337+3	182	.56941319+3
33	.10131364+3	83	.25839493+3	133	.41547496+3	183	.57255477+3
34	.10445532+3	84	.26153653+3	134	.41861656+3	184	.57569637+3
35	.10759699+3	85	.26467814+3	135	.42175816+3	185	.57883797+3
36	.11073866+3	86	.26781974+3	136	.42489976+3	186	.58197956+3
37	.11388031+3	87	.27096134+3	137	.42804135+3	187	.58512115+3
38	.11702197+3	88	.27410295+3	138	.43118295+3	188	.58826275+3
39	.12016363+3	89	.27724455+3	139	.43432455+3	189	.59140434+3
40	.12330528+3	90	.28038616+3	140	.43746614+3	190	.59454595+3
41	.12644693+3	91	.28352776+3	141	.44060774+3	191	.59768753+3
42	.12958857+3	92	.28666937+3	142	.44374934+3	192	.60082913+3
43	.13273022+3	93	.28981097+3	143	.44689094+3	193	.60397072+3
44	.13587186+3	94	.29295257+3	144	.45003253+3	194	.60711232+3
45	.13901349+3	95	.29609417+3	145	.45317413+3	195	.61025392+3
46	.14215513+3	96	.29923578+3	146	.45631573+3	196	.61339550+3
47	.14529676+3	97	.30237738+3	147	.45945732+3	197	.61653710+3
48	.14843839+3	98	.30551898+3	148	.46259892+3	198	.61967870+3
49	.15158003+3	99	.30866058+3	149	.46574052+3	199	.62282030+3
50	.15472165+3	100	.31180219+3	150	.46888211+3	200	.62596190+3

NOTE: A group of digits followed by +m indicates that the decimal point should be m places to the right of the first digit.

TABLE III.- THE ROOTS λ_n/H_i FOR $\alpha_i/H_i = -1.0$

n	λ_n/H_i	n	λ_n/H_i	n	λ_n/H_i	n	λ_n/H_i
1	0.12557837-1	51	0.15786898+3	101	0.31494664+3	151	0.47202561+3
2	.40794776-1	52	.16101050+3	102	.31808822+3	152	.47516719+3
3	.71557990+1	53	.16415202+3	103	.32122979+3	153	.47830878+3
4	.10270984+2	54	.16729354+3	104	.32437136+3	154	.48145036+3
5	.13398397+2	55	.17043506+3	105	.32751293+3	155	.48459195+3
6	.16531158+2	56	.17357659+3	106	.33065451+3	156	.48773354+3
7	.19666727+2	57	.17671812+3	107	.33379608+3	157	.49087512+3
8	.22803950+2	58	.17985965+3	108	.33693766+3	158	.49401670+3
9	.25942228+2	59	.18300118+3	109	.34007924+3	159	.49715829+3
10	.29081221+2	60	.18614271+3	110	.34322082+3	160	.50029986+3
11	.32220720+2	61	.18928425+3	111	.34636239+3	161	.50344145+3
12	.35360590+2	62	.19242579+3	112	.34950396+3	162	.50658304+3
13	.38500742+2	63	.19556733+3	113	.35264554+3	163	.50972462+3
14	.41641110+2	64	.19870888+3	114	.35578711+3	164	.51286621+3
15	.44781650+2	65	.20185041+3	115	.35892869+3	165	.51600780+3
16	.47922328+2	66	.20499197+3	116	.36207027+3	166	.51914938+3
17	.51063119+2	67	.20813351+3	117	.36521185+3	167	.52229096+3
18	.54204002+2	68	.21127506+3	118	.36835343+3	168	.52543256+3
19	.57344964+2	69	.21441661+3	119	.37149501+3	169	.52857413+3
20	.60485990+2	70	.21755816+3	120	.37463658+3	170	.53171572+3
21	.63627072+2	71	.22069971+3	121	.37777817+3	171	.53485731+3
22	.66768204+2	72	.22384126+3	122	.38091975+3	172	.53799889+3
23	.69909374+2	73	.22698282+3	123	.38406132+3	173	.54114048+3
24	.73050584+2	74	.23012437+3	124	.38720290+3	174	.54428206+3
25	.76191823+2	75	.23326593+3	125	.39034448+3	175	.54742365+3
26	.79333092+2	76	.23640748+3	126	.39348606+3	176	.55056524+3
27	.82474384+2	77	.23954904+3	127	.39662764+3	177	.55370682+3
28	.85615698+2	78	.24269060+3	128	.39976922+3	178	.55684841+3
29	.88757031+2	79	.24583216+3	129	.40291080+3	179	.55999000+3
30	.91898385+2	80	.24897373+3	130	.40605239+3	180	.56313158+3
31	.95039752+2	81	.25211528+3	131	.40919396+3	181	.56627316+3
32	.98181134+2	82	.25525685+3	132	.41233555+3	182	.56941476+3
33	.10132252+3	83	.25839841+3	133	.41547713+3	183	.57255635+3
34	.10446393+3	84	.26153997+3	134	.41861871+3	184	.57569793+3
35	.10760535+3	85	.26468154+3	135	.42176028+3	185	.57883951+3
36	.11074678+3	86	.26782310+3	136	.42490187+3	186	.58198110+3
37	.11388822+3	87	.27096467+3	137	.42804345+3	187	.58512269+3
38	.11702966+3	88	.27410623+3	138	.43118503+3	188	.58826428+3
39	.12017111+3	89	.27724779+3	139	.43432662+3	189	.59140586+3
40	.12331258+3	90	.28038937+3	140	.43746819+3	190	.59454746+3
41	.12645404+3	91	.28353093+3	141	.44060978+3	191	.59768904+3
42	.12959551+3	92	.28667250+3	142	.44375136+3	192	.60083062+3
43	.13273700+3	93	.28981407+3	143	.44689295+3	193	.60397221+3
44	.13587847+3	94	.29295564+3	144	.45003453+3	194	.60711379+3
45	.13901997+3	95	.29609721+3	145	.45317611+3	195	.61025538+3
46	.14216146+3	96	.29923879+3	146	.45631769+3	196	.61339697+3
47	.14530296+3	97	.30238035+3	147	.45945928+3	197	.61653856+3
48	.14844446+3	98	.30552193+3	148	.46260086+3	198	.61968015+3
49	.15158597+3	99	.30866349+3	149	.46574244+3	199	.62282173+3
50	.15472747+3	100	.31180507+3	150	.46888403+3	200	.62596333+3

NOTE: A group of digits followed by +m indicates that the decimal point should be m places to the right of the first digit.

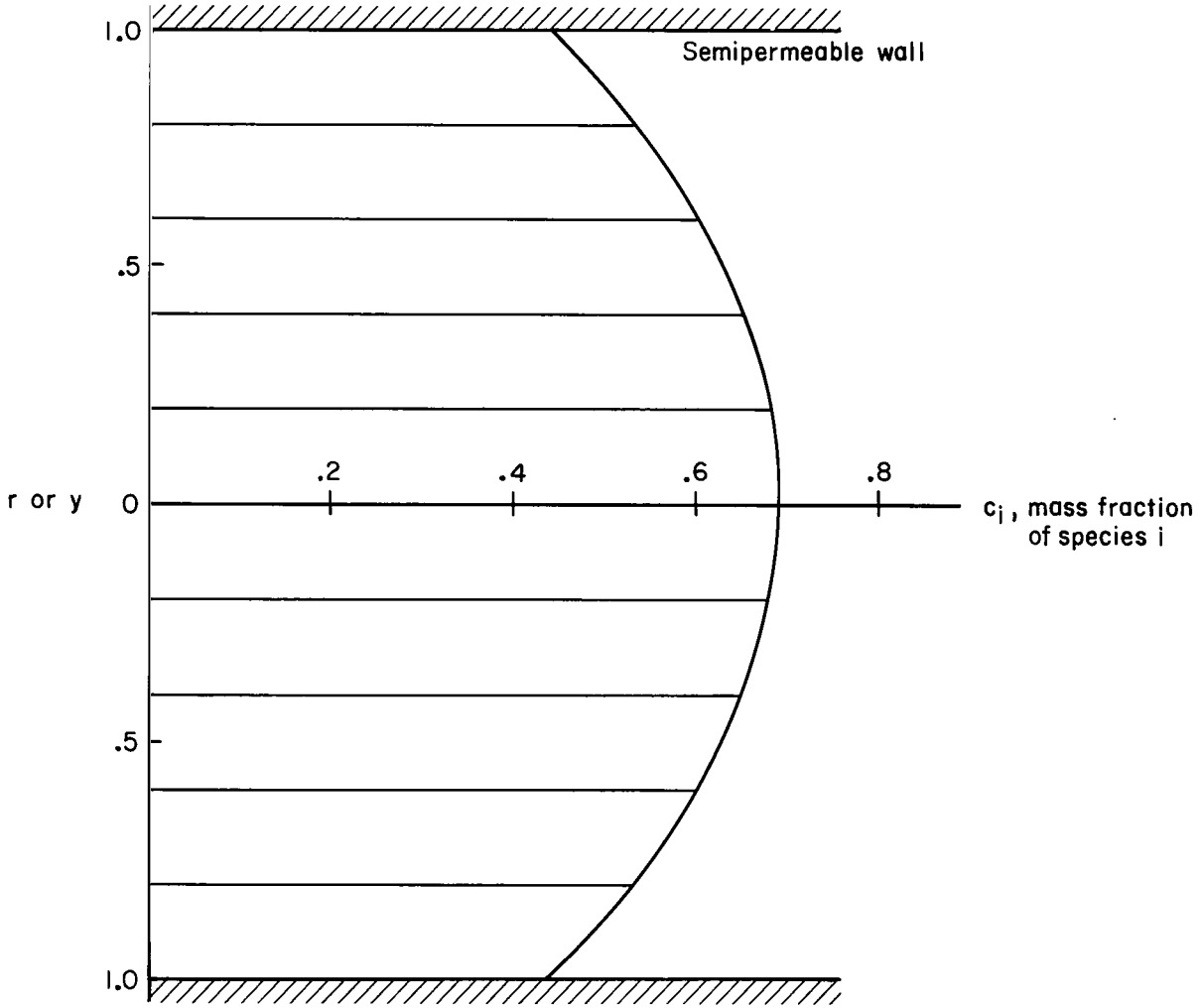

Figure 1.- Mass fraction profile for steady state and constant production ($H_i^2 = p^2 = 1$, $\alpha_i = -8$, $\beta_i = -3$).

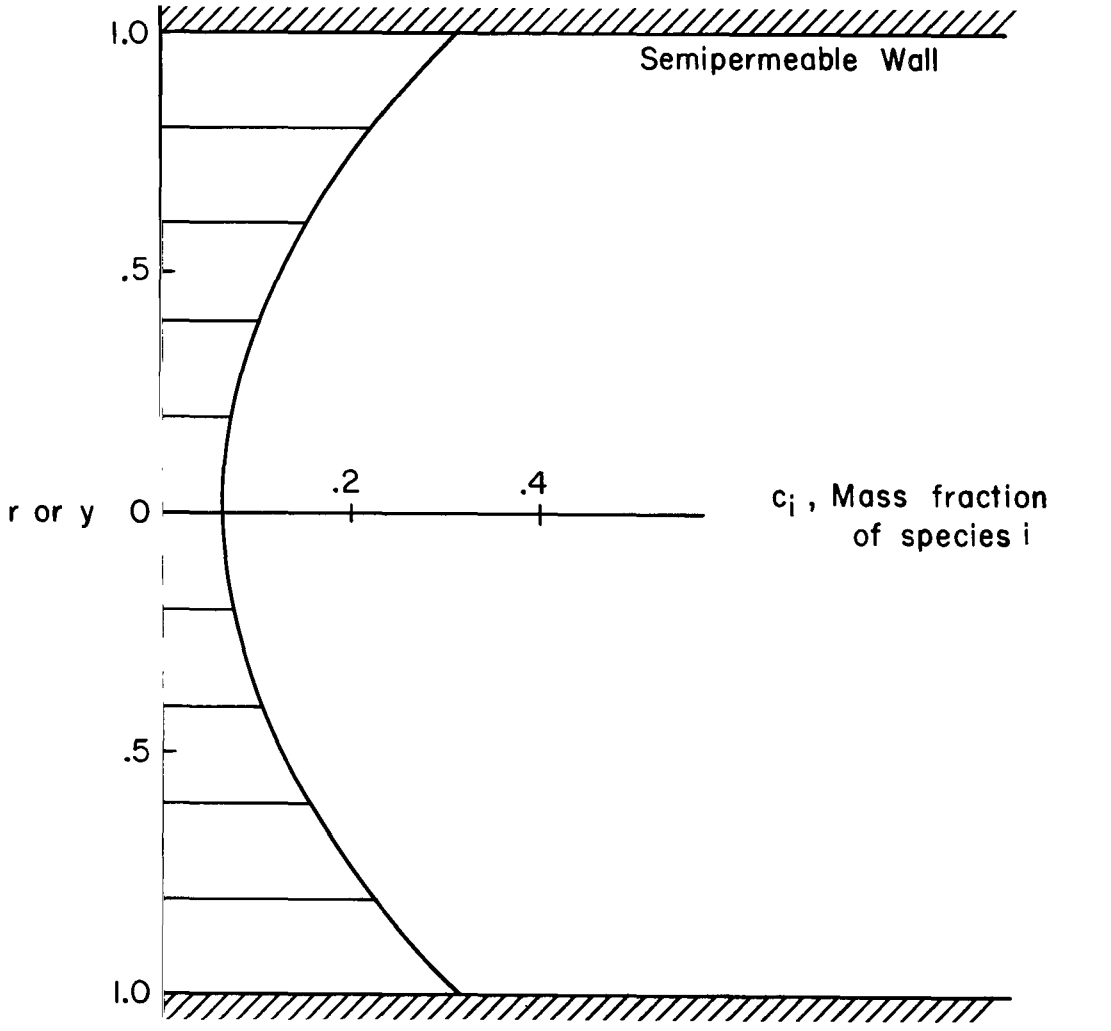

Figure 2.- Mass fraction profile for steady state and constant consumption ($H_i^2 = -p^2 = 1$, $\alpha_i = -8$, $\beta_i = -3$).

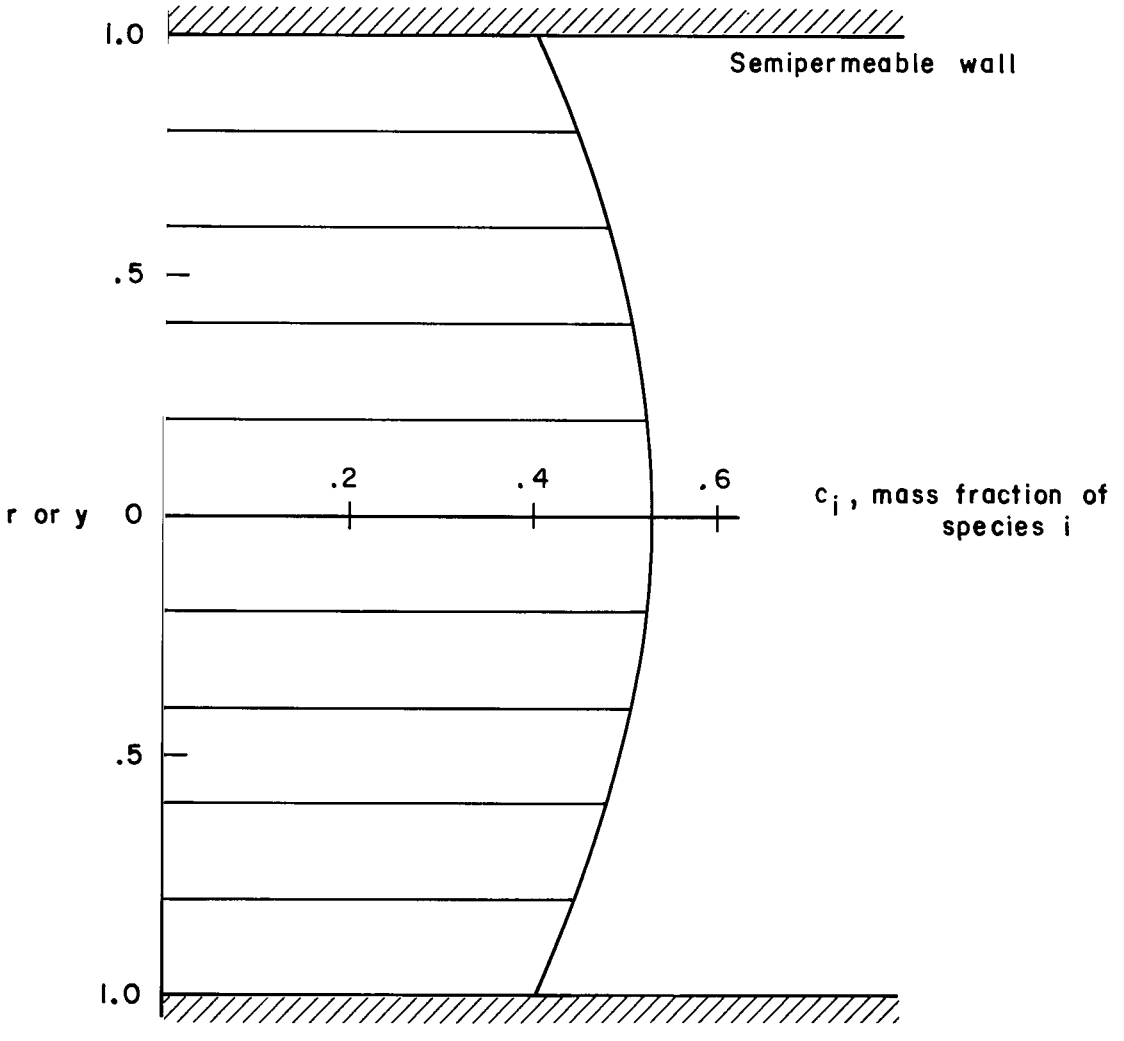

Figure 3.- Mass fraction profile for steady state and production proportional to concentration ($H_i^2 = p^2 = 1$, $\alpha_i = -8$, $\beta_i = -3$).

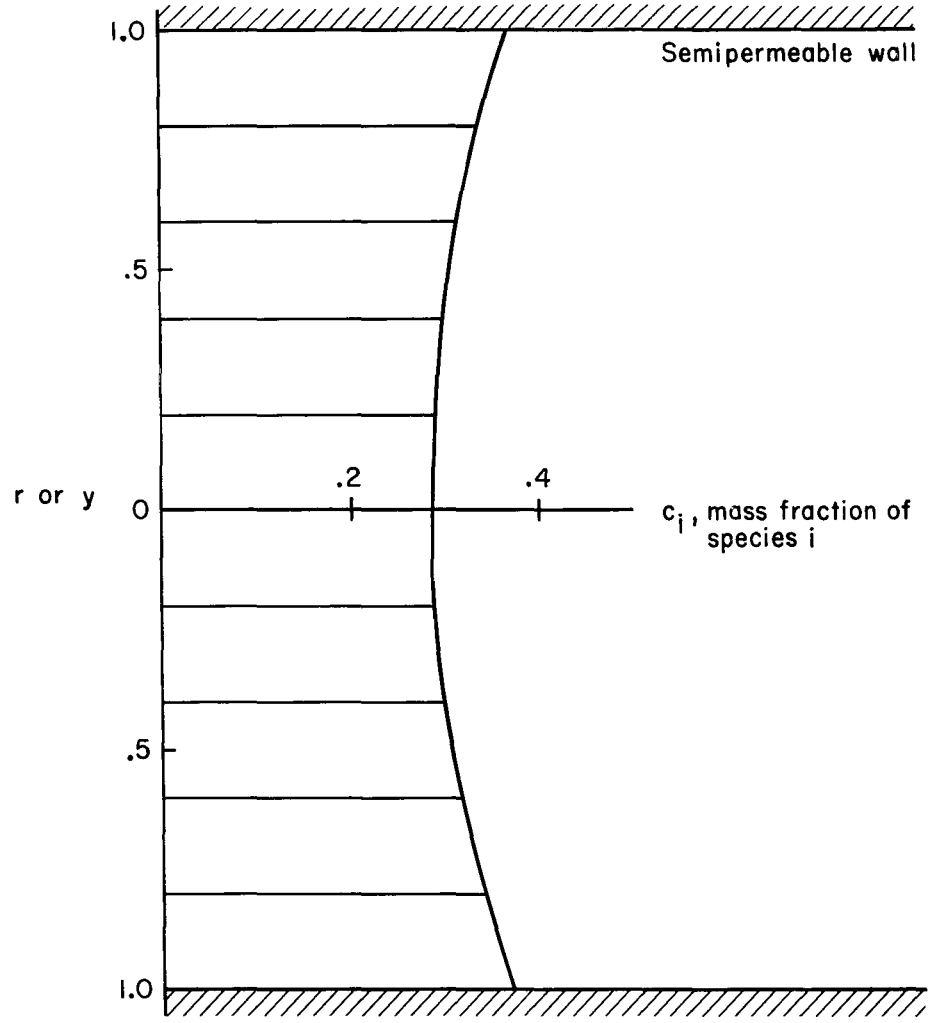

Figure 4.- Mass fraction profiles for steady state and consumption proportional to concentration ($H_i^2 = -p^2 = 1$, $\alpha_i = -8$, $\beta_i = -3$).

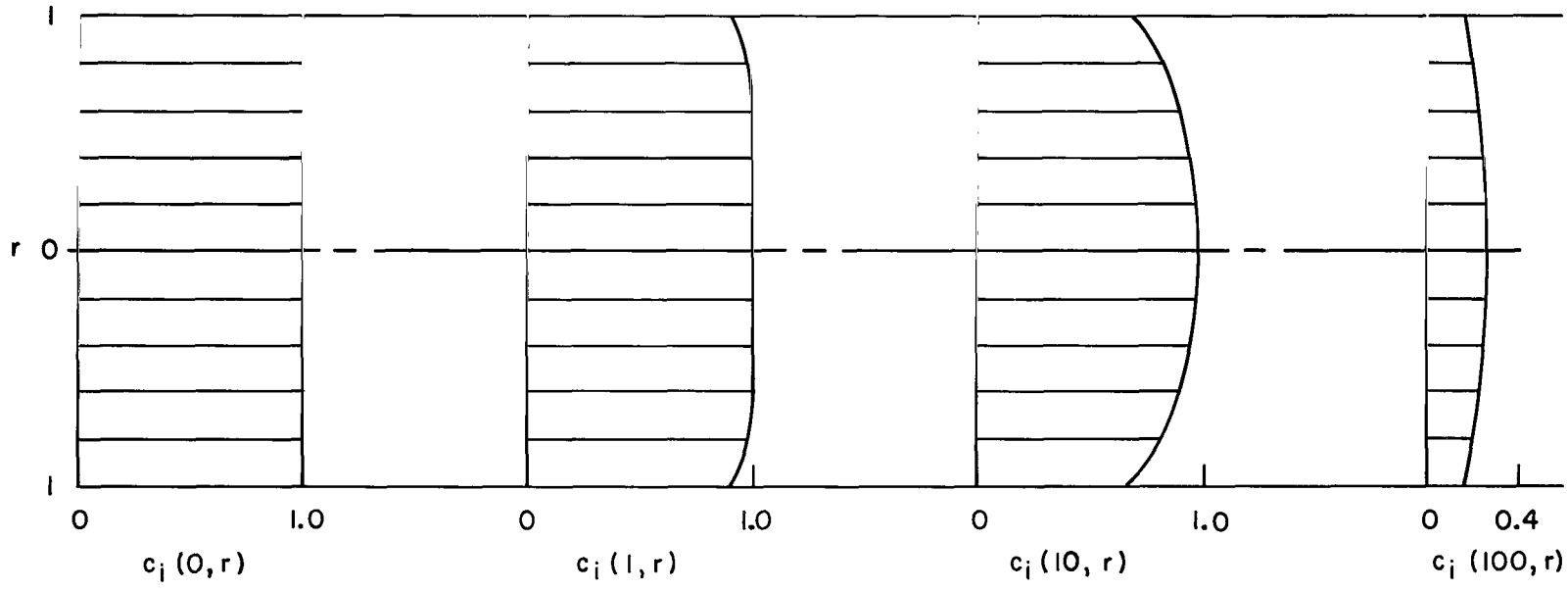

Figure 5.- Development of mass fraction profiles for steady-state bulk flow with no reactions ($H_i = -\alpha_i = 0.1$, $\beta_i = p^2 = 0$).

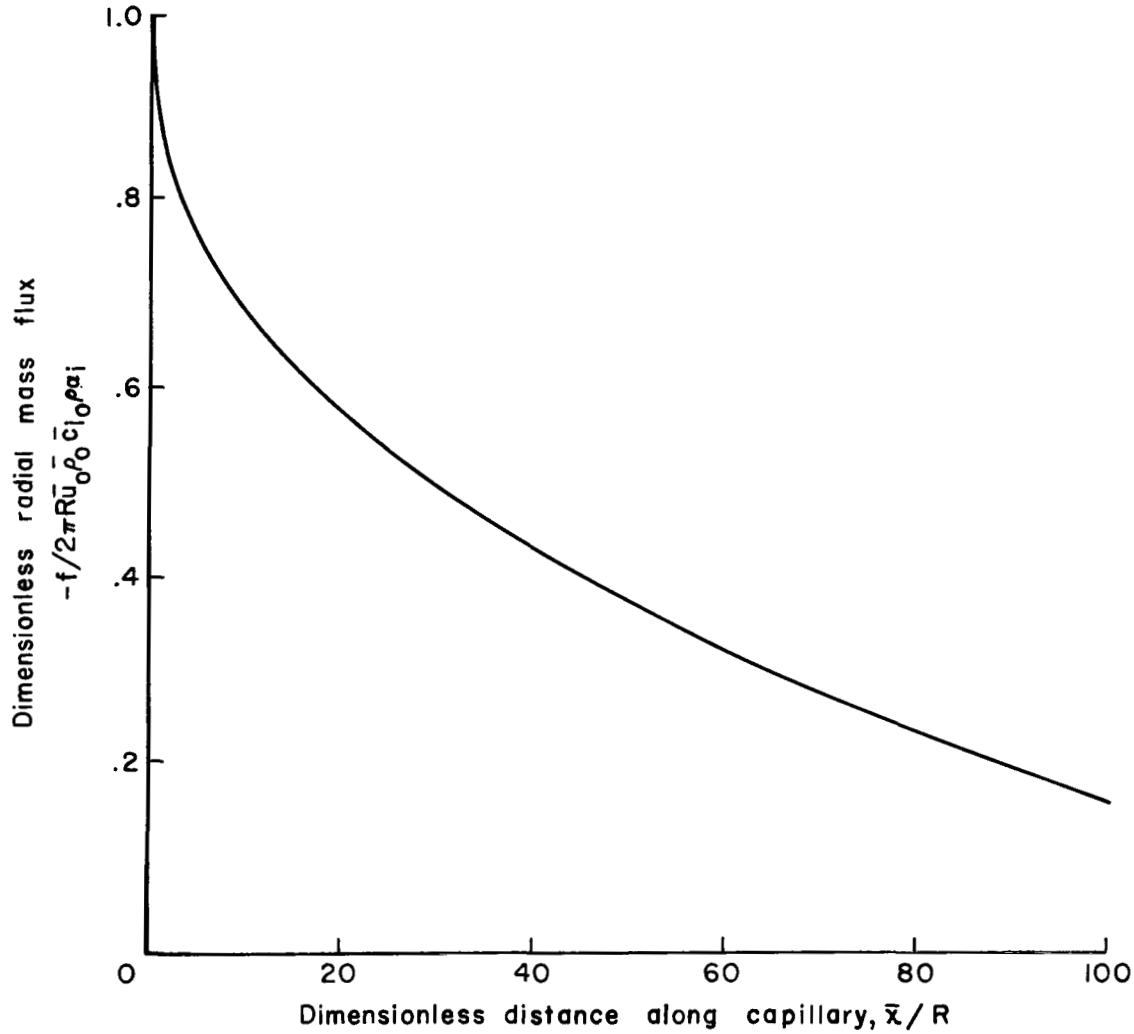

Figure 6.- Dimensionless mass flux per unit length of capillary leaving the semipermeable wall for steady-state bulk flow with no reactions ($H_i = -\alpha_i = 0.1$, $\beta_i = p^2 = 0$).

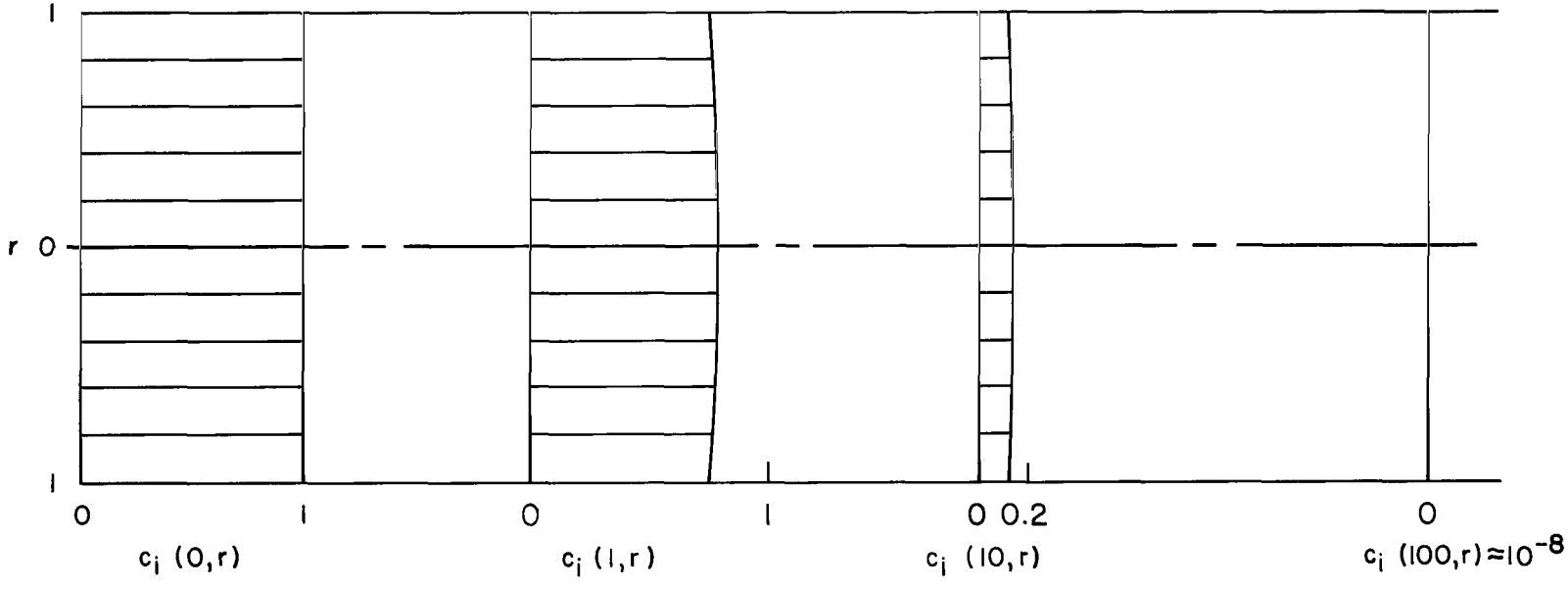

Figure 7.- Development of mass fraction profile for unsteady state, diffusion controlling with no reactions ($H_i = 1.0$, $\alpha_i = -0.1$, $\beta_i = p^2 = 0$).

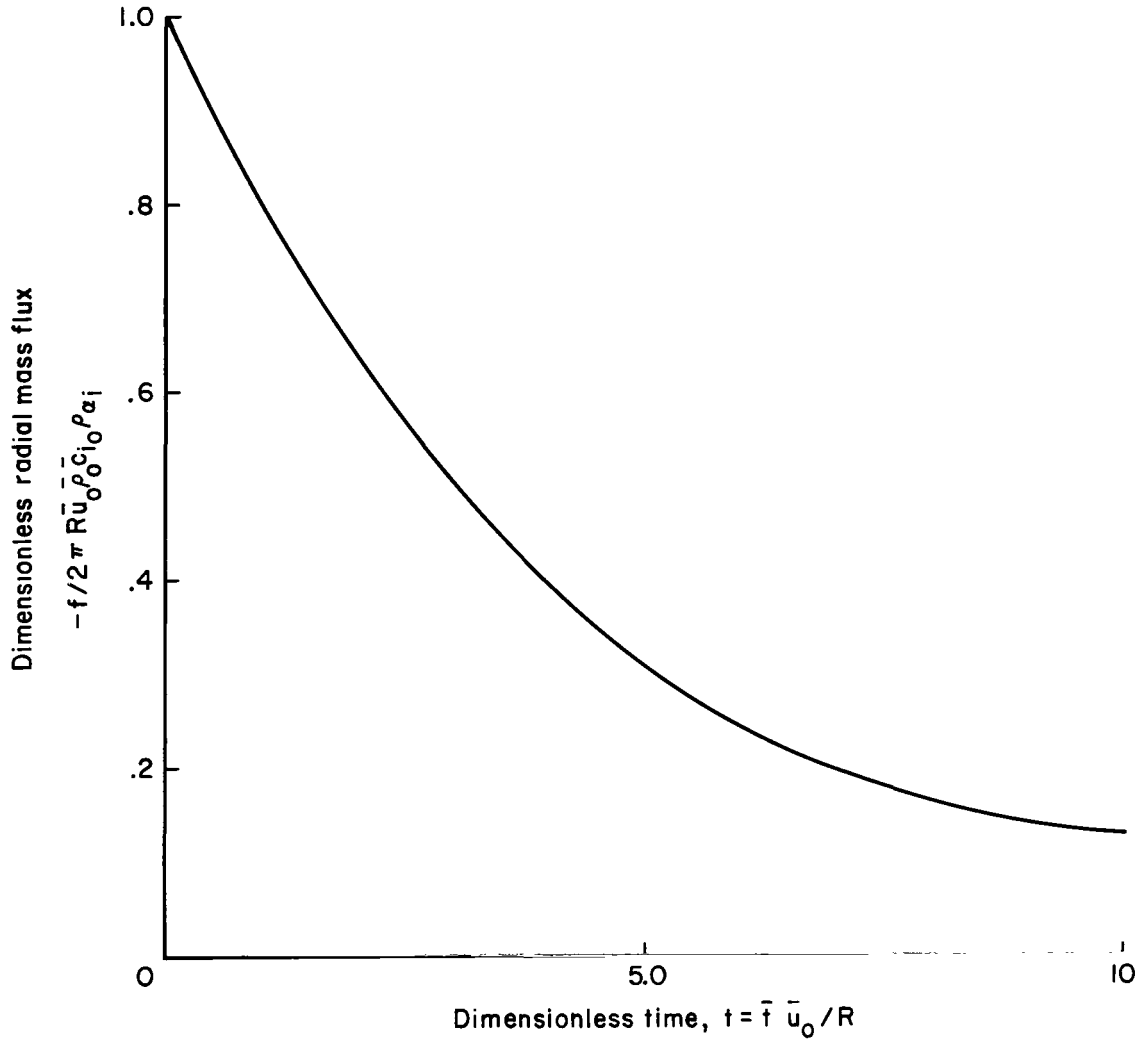

Figure 8.- Dimensionless mass flux per unit length of capillary leaving the semipermeable wall for unsteady state, diffusion controlling with no reactions ($H_i = 1.0$, $\alpha_i = -0.1$, $\beta_i = p^2 = 0$).

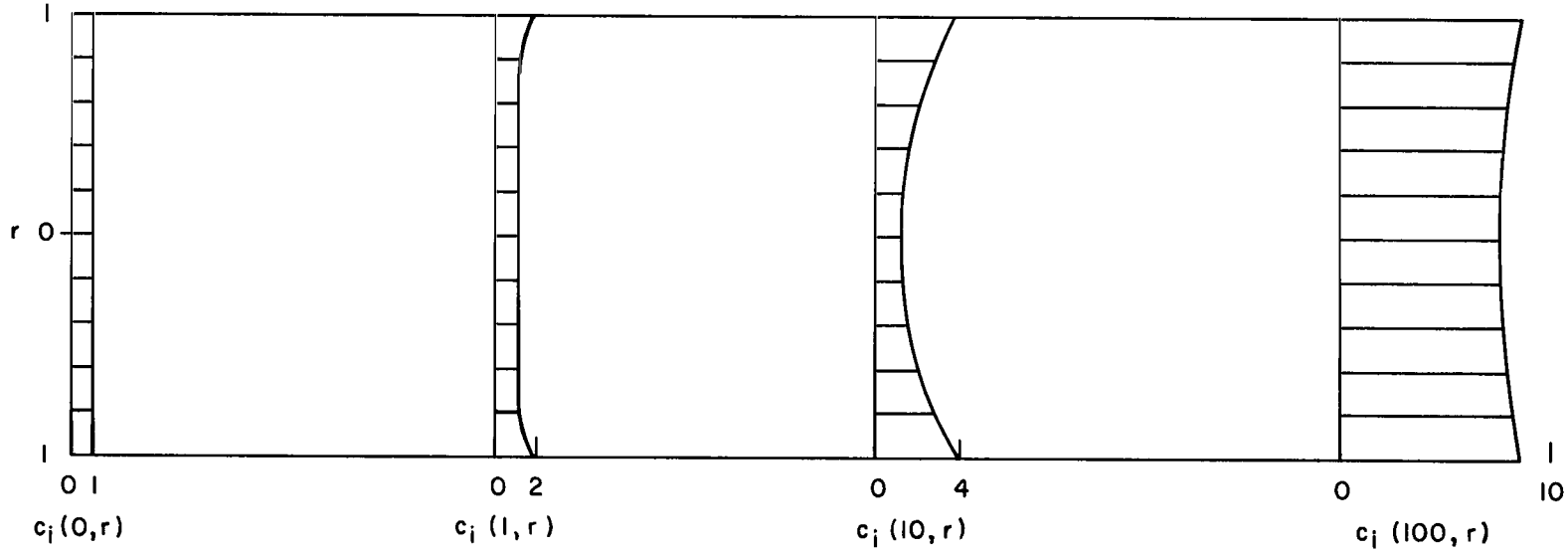

Figure 9.- Development of mass fraction profile for steady-state bulk flow with no reactions ($H_i = -\alpha_i = 0.1$, $\beta_i = -1.0$, $p^2 = 0$).

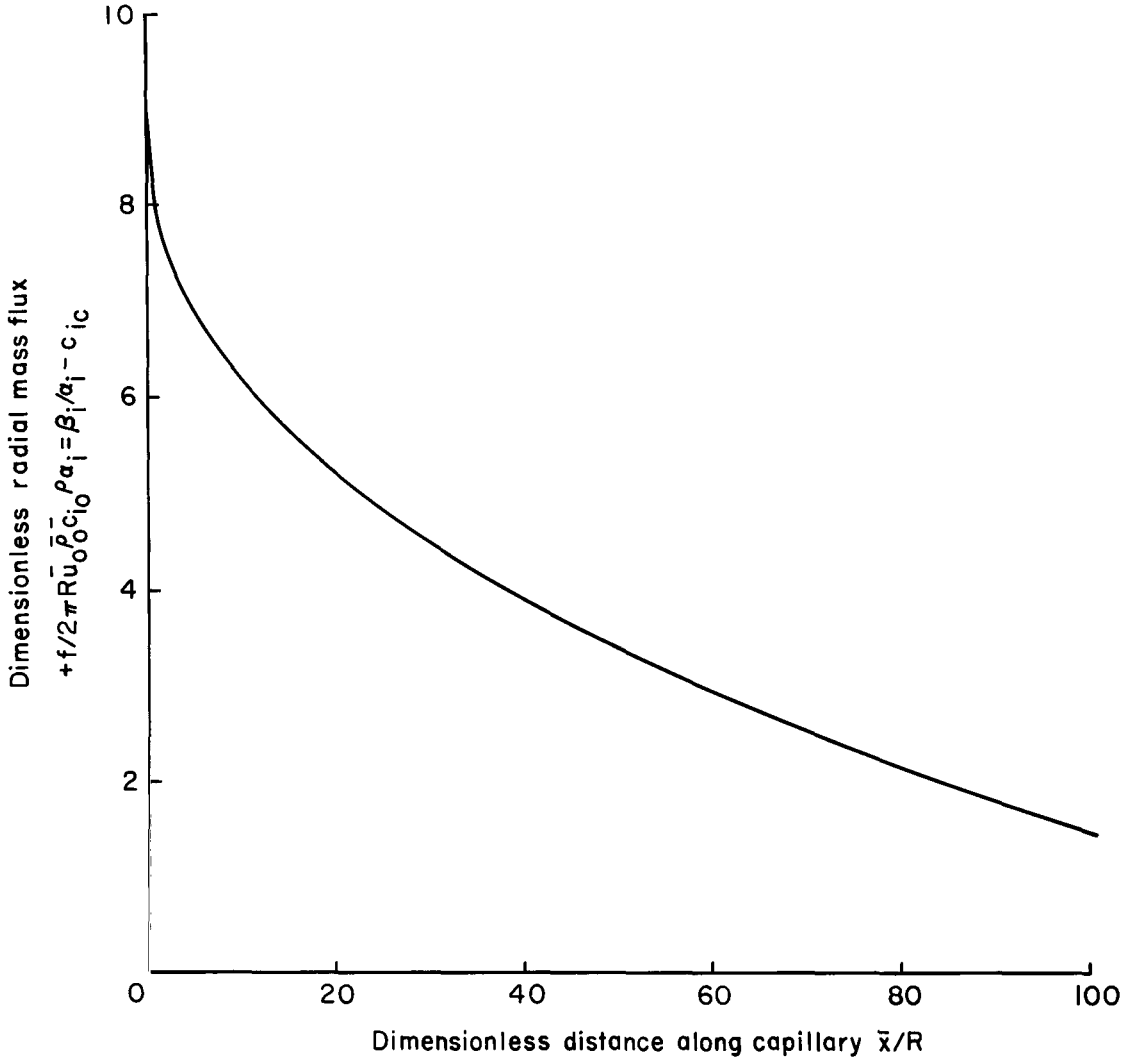

Figure 10.- Dimensionless mass flux per unit length of capillary entering the semipermeable wall for steady-state bulk flow with no reactions ($H_i = -\alpha_i = 0.1$, $\beta_i = -1.0$, $p^2 = 0$).

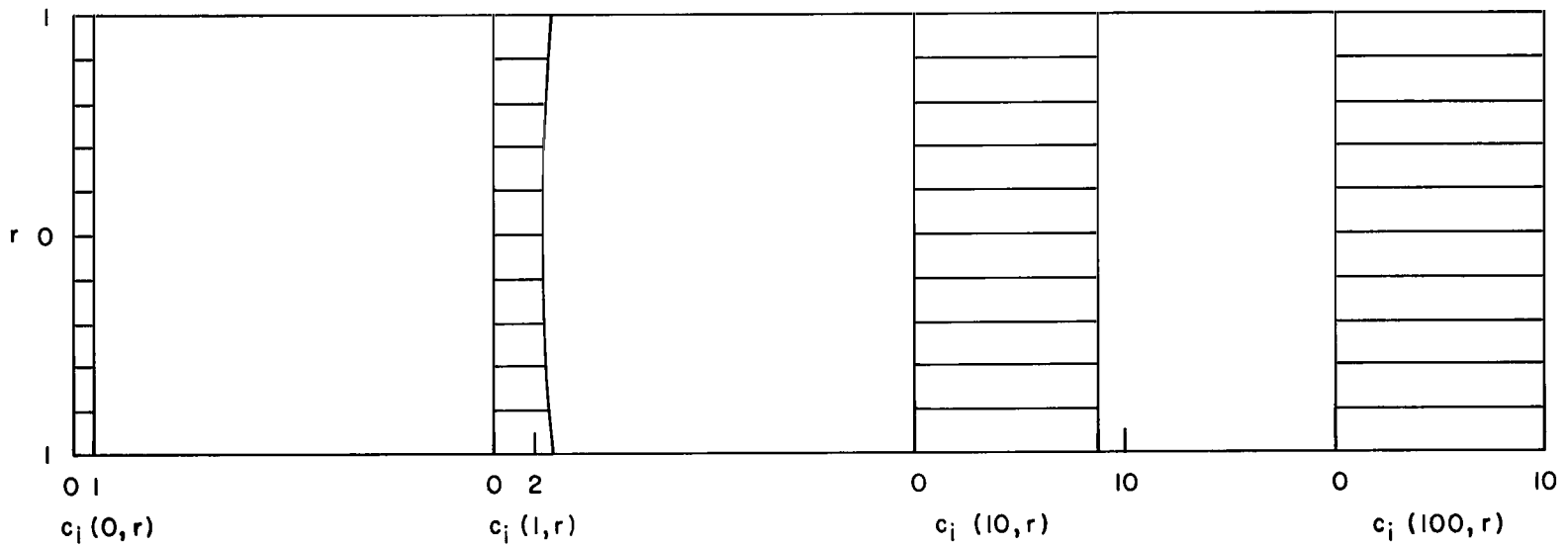

Figure 11.- Development of mass fraction profile for unsteady state, diffusion controlling with no reactions ($H_i = 1.0$, $\alpha_i = -0.1$, $\beta_i = -1.0$, $p^2 = 0$).

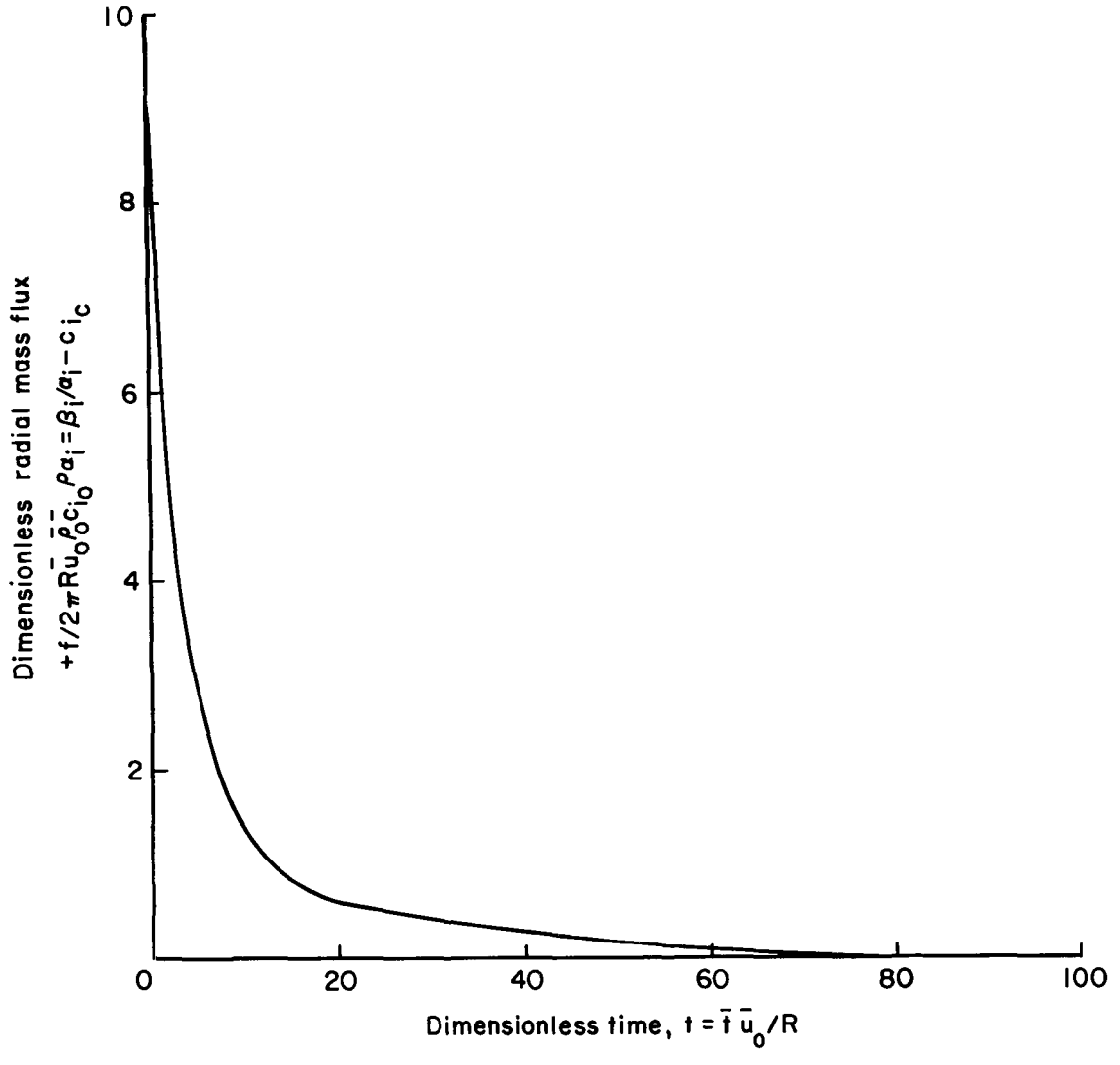

Figure 12.- Dimensionless mass flux per unit length of capillary entering the semipermeable wall for unsteady state, diffusion controlling with no reactions ($H_i = 1.0$, $\alpha_i = -0.1$, $\beta_i = -1.0$, $p^2 = 0$).